Obstructions for quantitative measure equivalence between locally compact groups

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Abstract

Given a measure equivalence coupling between two finitely generated groups, Delabie, Koivisto, Le Maître and Tessera have found explicit upper bounds on how integrable the associated cocycles can be. We extend these results to the broader framework of unimodular compactly generated locally compact groups. We also generalize a result by the first-named author, showing that the integrability threshold described in these statements cannot be achieved.

Keywords: Quantitative measure equivalence, locally compact groups, isoperimetric profile, volume growth.

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Contents

1 Introduction

Gromov introduced measure equivalence for countable groups as a measurable analogue of quasi-isometry.

Definition 1.1. Two countable groups Γ and Λ are *measure equivalent* if there exists a standard measured space (Ω, μ) equipped with commuting measure-preserving Γ- and Λ-actions such that

1. both actions are free;
2. the Γ- and Λ-actions admit Borel fundamental domains X_Γ and X_Λ respectively;
3. X_Γ and X_Λ have finite measure.

The quadruple $(\Omega, X_\Gamma, X_\Lambda, \mu)$ is called a *measure equivalence coupling* between Γ and Λ. It is straightforward to prove that $\mu(X_\Gamma)$ and $\mu(X_\Lambda)$ are positive, and the restrictions of μ to the fundamental domains X_Γ and X_Λ are respectively denoted by ν_Γ and ν_Λ.

The most natural example to keep in mind is that of lattices in the same locally compact group. Another instance of measure equivalence comes from ergodic theory.

Definition 1.2 (Dye [Dye59])**.** Two countable groups Γ and Λ are *orbit equivalent* if there exist free probability measure-preserving Γ- and Λ-actions on a standard probability space (X, μ) such that for almost every $x \in X$, $\Gamma \cdot x = \Lambda \cdot x$.

Two groups are orbit equivalent if and only if they are measure equivalent with common fundamental domains (see [Fur11] for a proof).

This paper deals with measure equivalence in the setting of locally compact groups, a precise definition in this context will be given in Section 2.2. Measure and orbit equivalences have already been studied for such groups, see e.g. [BFS13; CLM17; KKR21b; KKR21a; Pau24; DLIT25], and it is worth mentioning that measure equivalence and orbit equivalence are the same notions among non-discrete, locally compact, second countable groups [KKR21a, Theorem A]. Before stating our main results, we briefly review the discrete case

Many results have been obtained for countable non-amenable groups, most of which describe rigidity phenomena. We refer the reader to the works of

- Furman on lattices in higher rank semi-simple Lie groups [Fur99];
- Kida on mapping class groups [Kid08; Kid10];
- Guirardel and Horbez on $\mathrm{Out}(F_N)$ [GH21];
- Horbez and Huang on right-angled Artin groups [HH22; HH24];
- Escalier and Horbez on graph products [EH24].

However, Ornstein and Weiss proved that any two free probability measure-preserving actions of infinite amenable groups are orbit equivalent [OW80]. In particular, measure equivalence is trivial for such groups. In the non-discrete setting, Koivisto, Kyed and Raum proved that the class of amenable, locally compact, second countable consists in three measure equivalence classes: compact groups, non-compact unimodular amenable groups and non-unimodular amenable groups [KKR21a, Theorem B]. To counter this flexibility phenomenon, we impose restrictions on the functions arising from measure equivalence couplings, namely the *cocycles*.

Definition 1.3. Let $(\Omega, X_\Gamma, X_\Lambda, \mu)$ be a measure equivalence coupling between discrete groups Γ and Λ. For every $\gamma \in \Gamma$ and $\lambda \in \Lambda$, and for almost every $x_\Gamma \in X_\Gamma$ and $x_\Lambda \in X_\Lambda$, there exist unique $\alpha(\gamma, x_\Lambda) \in \Lambda$ and $\beta(\lambda, x_\Gamma) \in \Gamma$ such that

$$\alpha(\gamma, x_\Lambda) * \gamma * x_\Lambda \in X_\Lambda \text{ and } \beta(\lambda, x_\Gamma) * \lambda * x_\Gamma \in X_\Gamma,$$

where uniqueness follows from the fact that X_Γ and X_Λ are fundamental domains for the Γ- and the Λ-actions respectively. The measurable maps $\alpha\colon \Gamma \times X_\Lambda \to \Lambda$ and $\beta\colon \Lambda \times X_\Gamma \to \Gamma$ are the *cocycles* associated to this coupling.

We now assume that Γ and Λ are finitely generated groups, with finite generating subsets S_Γ and S_Λ, and we denote by $|.|_{S_\Gamma}$ and $|.|_{S_\Lambda}$ the associated word metric.

Given $p, q \in [0, +\infty]$, an $(\mathrm{L}^p, \mathrm{L}^q)$ *measure equivalence coupling* from Γ to Λ is a measure equivalence coupling such that for every $\gamma \in \Gamma$, the map $|\alpha(\gamma, .)|_{S_\Lambda}\colon X_\Lambda \to \mathbb{N}$ is L^p, and for every $\lambda \in \Lambda$, the map $|\beta(\lambda, .)|_{S_\Gamma}\colon X_\Gamma \to \mathbb{N}$ is L^q. In particular, L^0 means that there is no requirement on the corresponding cocycle. We can more generally define the notion of (φ, ψ)*-integrability* for any maps $\varphi, \psi\colon \mathbb{R}_+ \to \mathbb{R}_+$. We will give the definition in Section 2.2, where we define measure equivalence in the more general setting of unimodular locally compact second countable (lcsc) groups. For instance, L^p is exactly φ-integrability when $\varphi(t) = t^p$.

In [Sha04], Shalom was the first to study quantitative measure equivalence, more precisely the case of L^∞ measure equivalence, for amenable groups. His goal was to prove the quasi-isometry invariance of Betti numbers for nilpotent groups. We say that a measure equivalence coupling $(\Omega, X_\Gamma, X_\Lambda, \mu)$ is *mutually cobounded* if there exist finite subsets $F_\Lambda \subset \Lambda$ and $F_\Gamma \subset \Gamma$ such that $X_\Gamma \subset F_\Lambda \cdot X_\Lambda$ and $X_\Lambda \subset F_\Gamma \cdot X_\Gamma$.

Theorem 1.4 ([Sha04])**.** *Two countable amenable groups are quasi-isometric if and only if there exists an* L^∞ *mutually cobounded measure equivalence coupling between them.*

Historically, the notion of L^p measure equivalence, for $p \geqslant 1$, was introduced in [BFS13], and more generally (φ, ψ)-integrable measure equivalence was first defined in [DKLMT22] to study the weaker notion of L^p orbit equivalence for $p < 1$. For weaker assumptions than L^∞, which geometric properties of amenable groups can be captured? In [Aus16], Austin first proved that integrably measure equivalent groups of polynomial growth have bi-Lipschitz equivalent asymptotic cones.

Behaviour of volume growth. Many finer rigidity results have been obtained, involving the volume growth. Given a finitely generated group Γ, with a finite generating subset S_Γ, the *volume growth* with respect to S_Γ is the map $V_{S_\Gamma}\colon \mathbb{N} \to \mathbb{N}$ defined by

$$V_{S_\Gamma}(n) = |\{\gamma \in \Gamma \mid |\gamma|_{S_\Gamma} \leqslant n\}|.$$

It is straightforward to show that two different finite generating sets yield asymptotically equivalent growth functions, and the common asymptotic behaviour is denoted by V_Γ.

For instance, $V_{\mathbb{Z}^d}(n) \simeq n^d$. More generally, a celebrated result of Gromov [Gro81] states that virtually nilpotent groups are exactly the groups with polynomial growth. Given non-trivial groups F and Γ, we define the wreath product $F \wr \Gamma$ as

$$F \wr \Gamma = \left(\bigoplus_{\Gamma} F \right) \rtimes \Gamma$$

where the action of Γ on the direct sum is induced by the action of Γ on itself by left translation. It is well known that if F and Γ are finitely generated, then so is $F \wr \Gamma$. If moreover F is not trivial and Γ is infinite, then $V_{F \wr \Gamma}(n) \simeq e^n$.

The notion of volume growth also exists in the case of compactly generated locally compact groups, see Section 2.1.

In the appendix of [Aus16], Bowen proved an obstruction for integrable measure equivalence, using the notion of volume growth of finitely generated groups.

Theorem 1.5 ([Aus16, Theorem B.2]). *Let Γ and Λ be finitely generated groups. If Γ and Λ are L^1 measure equivalent, then $V_\Gamma(n) \simeq V_\Lambda(n)$.*

For instance, $\mathbb{Z}^k$ and $\mathbb{Z}^d$ are integrably measure equivalent if and only if $k = d$. The results of Austin and Bowen show that integrable measure equivalent preserves significant coarse geometric invariants. It is therefore natural to wonder whether these rigidity results still hold for more general quantifications. In the wider setup of (φ, ψ)-integrability, Delabie, Koivisto, Le Maître and Tessera refined Bowen's result as follows.

Theorem 1.6 ([DKLMT22, Theorem 3.1]). *Let Γ and Λ be finitely generated groups. Let $\varphi\colon \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be an increasing and subadditive map. If there is a (φ, L^0)-integrable measure equivalence coupling from Γ to Λ, then*

$$V_\Gamma(n) \preccurlyeq V_\Lambda(\varphi^{-1}(n)).$$

For instance, given positive integers k and ℓ, there is no $(\mathrm{L}^p, \mathrm{L}^0)$ measure equivalence coupling from $\mathbb{Z}^{k+\ell}$ and $\mathbb{Z}^k$ if $p > \frac{k}{k+\ell}$.

Delabie, Llosa Isenrich and Tessera have recently brought a first extension to locally compact groups when $\varphi(t) = t^p$.

Theorem 1.7 ([DLIT25, Theorem A.5]). *Let G and H be non-discrete unimodular locally compact compactly generated groups and let $p \in]0,1]$. If there exists an $(\mathrm{L}^p, \mathrm{L}^0)$-integrable measure equivalence coupling from G to H, then*

$$V_G(n) \preccurlyeq V_H(n^{1/p}).$$

We extend this result to (φ, L^0)-measure equivalence, where $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ is any increasing and subadditive map, for instance $\varphi(t) = \log(1+t)$.

Theorem A (see Theorem 4.1). *Let G and H be non-discrete unimodular locally compact compactly generated groups and $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing and subadditive map. If there exists a (φ, L^0)-integrable measure equivalence coupling from G to H, then*

$$V_G(n) \preccurlyeq V_H(\varphi^{-1}(n)).$$

Remark 1.8. We focus on unimodular groups because φ-integrability is undefined in the non-unimodular setting. Indeed, a definition of measure equivalence exists in the non-unimodular case (see e.g. [KKR21b, Definition 3.2]), but asks for the more general notion of non-singular actions. Another obstruction for an extension to non-unimodular groups will be the fact that isoperimetric profile, that we now deal with, does not contain any interesting information in the case of non-unimodular locally compact groups (we refer the reader to [Tes13, comments before Section 5.2]).

Behaviour of isoperimetric profiles. A limitation of the previous rigidity results is that volume growth cannot distinguish between groups that get bigger and bigger. For instance, in the case of discrete groups, the following iterations of lamplighters

$$\mathbb{Z}/2\mathbb{Z} \wr (\mathbb{Z}/2\mathbb{Z} \wr \ldots (\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}) \ldots)$$

all have exponential growth, but it is natural to expect that such groups are integrably measure equivalent only if they have the same number of iterations. The isoperimetric profiles enable us to distinguish them.

For a finitely generated group Γ with a finite generating set S_Γ, and given $p \geqslant 1$, its ℓ^p*-isoperimetric profile* is the function $j_{p,\Gamma} \colon \mathbb{N} \longrightarrow \mathbb{R}_+$ given by

$$j_{p,\Gamma}(n) \coloneqq \sup_{\substack{f\colon \Gamma \to \mathbb{R}_+ \\ |\mathrm{supp} f| \leqslant n}} \frac{\|f\|_p}{\|\nabla_\Gamma f\|_p}$$

where the support of $f\colon \Gamma \longrightarrow \mathbb{R}_+$ is $\mathrm{supp} f \coloneqq \{\gamma \in \Gamma : f(\gamma) \neq 0\}$ and the ℓ^p-norm of its gradient is defined by

$$\|\nabla_\Gamma f\|_p^p \coloneqq \sum_{\gamma \in \Gamma,\ s \in S_\Gamma} \left| f(\gamma) - f(s^{-1}\gamma) \right|^p.$$

In the case $p = 1$, this function is simply called *isoperimetric profile*, and has a simpler definition (up to asymptotic equivalence), namely

$$j_{1,\Gamma}(n) \simeq \sup_{|A| \leqslant n} \frac{|A|}{|\partial_\Gamma A|},$$

where $\partial_\Gamma A \coloneqq A S_\Gamma \backslash A = \{\gamma \in \Gamma \backslash A : \exists s \in S_\Gamma, \exists a \in A, \gamma = as\}$ is the *boundary* of A in Γ. It is well known that $j_{1,\Gamma}$ is the generalized inverse of the *Følner function*

$$\mathrm{Føl}_\Gamma(k) = \inf \left\{ |A| \colon \frac{|\partial A|}{|A|} \leqslant 1/k \right\}$$

first mentioned by Vershik [Ver73]. Finally a well-known inequality due to Coulhon and Saloff-Coste [CS93] relates $j_{1,\Gamma}$ and V_Γ:

$$j_{1,\Gamma}(n) \preccurlyeq V_\Gamma^{-1}(n),$$

where V_Γ^{-1} denotes the generalized inverse of the volume growth function. Here are examples of isoperimetric profiles for some groups:

- $j_{p,\Gamma}(n) \simeq n^{\frac{1}{d}}$ if Γ has polynomial growth of degree $d \geqslant 1$;
- $j_{p,\Gamma}(n) \simeq \ln(n)$ for the Baumslag-Solitar group $\Gamma = \mathrm{BS}(1,k)$ for $k \geqslant 2$, or $\Gamma = F \wr \mathbb{Z}$, where F is a non-trivial finite group;
- $j_{p,\Gamma}(n) \simeq \ln(n)$ for any polycyclic group Γ with exponential growth [Pit95; Pit00];
- $j_{1,F\wr\Gamma}(n) \simeq (\ln(n))^{\frac{1}{d}}$ with F finite, and Γ having polynomial growth of degree $d \geqslant 1$ [Ers03];
- Brieussel and Zheng [BZ21, Theorem 1.1] give a large description of asymptotic behaviours for isoperimetric profiles. For any non-decreasing function $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that $t \mapsto \frac{t}{f(t)}$ is non-decreasing, there exists a finitely generated group Γ with exponential volume growth having isoperimetric profile $j_{p,\Gamma}(n) \simeq \frac{\ln(n)}{f(\ln(n))}$.

Notice that the ℓ^p-isoperimetric profile of a finitely generated group is bounded if and only if the group is non-amenable. Moreover its asymptotic behaviour is, somehow, a measurement of its amenability; the faster it goes to infinity, the "more amenable" the group is. The following result clearly demonstrates that quantitative measure equivalence accurately captures the geometry of amenable groups.

Theorem 1.9 ([DKLMT22, Theorem 1.1])**.** *Let $\varphi\colon \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be a non-decreasing function. Let Γ and Λ be finitely generated groups. Assume that there exists a (φ, L^0)-integrable measure equivalence coupling from Γ to Λ. Then*

- *if $\varphi(t) = t^p$ with $p \geqslant 1$, then $j_{p,\Lambda}(n) \preccurlyeq j_{p,\Gamma}(n)$;*
- *if $t \mapsto \frac{t}{\varphi(t)}$ is non-decreasing, then $\varphi \circ j_{1,\Lambda}(n) \preccurlyeq j_{1,\Gamma}(n)$.*

In Section 2.3, we will introduce isoperimetric profiles in the setting of compactly generated locally compact unimodular groups.

Simplifying the proofs in [DKLMT22] for a more natural adaptation to the locally compact setting, we get the two following extensions of Theorem 1.9 to locally compact groups.

Theorem B (see Theorem 5.1)**.** *Let G and H be non-discrete unimodular locally compact compactly generated groups and $p \geqslant 1$. If there exists an $(\mathrm{L}^p, \mathrm{L}^0)$ measure equivalence coupling from G to H, then*

$$j_{p,H}(n) \preccurlyeq j_{p,G}(n).$$

Theorem C (see Theorem 5.2)**.** *Let G and H be non-discrete unimodular locally compact compactly generated groups and $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a non-decreasing map such that $t \mapsto t/\varphi(t)$ is non-decreasing. If there exists a (φ, L^0)-integrable measure equivalence coupling from G to H, then*

$$\varphi \circ j_{1,H}(n) \preccurlyeq j_{1,G}(n).$$

Remark 1.10. Delabie, Koivisto, Le Maître and Tessera [DKLMT22] also proved rigidity results for asymmetric notions of measure equivalence (measure subgroup, measure quotient, measure sub-quotient).

First, they find obstructions for quantitative versions, similar to Theorems 1.7 and 1.9. In the case of measure subgroups, these have been generalized to locally compact groups by the second-named author [Pau24]. In [DKLMT22], the authors need an assumption on the fundamental domains, "at most-m-to-one coupling", which becomes "coarsely m-to-one coupling" in [Pau24]. For instance, in the case of measure equivalence, these assumptions are implied by coboundedness. Thus we already knew from [Pau24] that Theorems B and C hold in the particular case of cobounded measure equivalence.

Secondly, in the same vein as Shalom's Theorem 1.4, the following is proven in [DKLMT22]: the existence of an at most m-to-one L^∞ measure subgroup from Γ to Λ is equivalent to the existence of a regular embedding $\Gamma \to \Lambda$. As a consequence of the quantitative results of [DKLMT22], the isoperimetric profiles are monotonous under regular embeddings. Once again, these results have been generalized to locally compact groups in [Pau24].

Main ingredients for the proof of Theorems A, B and C. In [DKLMT22], the authors crucially use the fact that any bijection between countable groups, or at least subsets of countable groups, is measure preserving. Indeed, the Haar measures are simply the counting measures in this context. These bijections are precisely the ones provided by the cocycles. This is the main issue when extending their main results for non-discrete locally compact groups since cocycles have no reason to provide measure-preserving maps between the groups.

In [DLIT25, Proposition A.1], the following is proven using cross-sections. Given a measure equivalence coupling between unimodular non-discrete locally groups, it is always possible to slightly modify the fundamental domains so that the cocycles satisfy the desired assumptions, namely providing Haar measure-preserving bijections between certain random measurable subsets of the groups.

Another hypothesis which will be useful for our proofs is ergodicity. To this end, we will use ideas from [KKR21a, Proposition 2.17 (ii)].

Further details are provided in Section 3.

Integrability threshold. Given positive integers $k, \ell \geqslant 1$ and polynomial growth groups Γ and Λ of degrees $k + \ell$ and k respectively, the obstructions provided in [DKLMT22] imply the following: there is no $(\mathrm{L}^p, \mathrm{L}^0)$ integrable measure equivalence coupling from Γ to Λ for any $p > \frac{k}{k+\ell}$. The existence of an $(\mathrm{L}^p, \mathrm{L}^0)$ integrable measure equivalence coupling for any $p < \frac{k}{k+\ell}$ have recently been proved in [DLIT25, Theorem 1.6]. For $\Gamma = \mathbb{Z}^{k+\ell}$ and $\Lambda = \mathbb{Z}^k$, the result was already known [DKLMT22, Theorem 1.9].

Now the question is the existence of such a coupling for the threshold $p = \frac{k}{k+\ell}$. For more general groups, not necessarily of polynomial growth, we would like to know if the bounds of integrability given by Theorems 1.6 and 1.9 can be reached. The first-named author has recently answered this question [Cor25, Theorems A, B and D] in the discrete setting and the following statements provide extensions to locally compact groups.

Theorem D (see Theorem 6.1)**.** *Let G and H be locally compact compactly generated groups. Assume that there exist a non-decreasing function f_G and an increasing function f_H satisfying $f_G(n) \simeq j_{1,G}(n)$, $f_H(n) \simeq j_{1,H}(n)$ and the following assumptions as $t \to +\infty$:*

$$f_G(t) = o\left(f_H(t)\right), \tag{1}$$

$$\forall C > 0,\ f_G(Ct) = O\left(f_G(t)\right), \tag{2}$$

$$\forall C > 0,\ f_G \circ f_H^{-1}(Ct) = O\left(f_G \circ f_H^{-1}(t)\right). \tag{3}$$

Then there is no $(f_G \circ f_H^{-1}, \mathrm{L}^0)$-integrable measure equivalence coupling from G to H.

Theorem E (see Theorem 6.2)**.** *Let G and H be locally compact compactly generated groups. Assume that there exist two increasing functions f_G and f_H satisfying $f_G(n) \simeq V_G(n)$, $f_H(n) \simeq V_H(n)$ and the following assumptions as $t \to +\infty$:*

$$f_G^{-1}(t) = o\left(f_H^{-1}(t)\right), \tag{4}$$

$$\forall C > 0,\ f_G^{-1}(Ct) = O\left(f_G^{-1}(t)\right), \tag{5}$$

$$\forall C > 0,\ f_G^{-1} \circ f_H(Ct) = O\left(f_G^{-1} \circ f_H(t)\right). \tag{6}$$

Then there is no $(f_G^{-1} \circ f_H, \mathrm{L}^0)$-integrable measure equivalence coupling from G to H.

Outline of the paper. After recalling some preliminaries in Section 2, we present in Section 3 a preparatory lemma for the main statements of the paper. The theorem on volume growth is proven in Section 4, the ones on isoperimetric profiles in Section 5, and Section 6 deals with the integrability thresholds.

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2 Preliminaries

2.1 Conventions and notations

In this paper, groups G and H are always assumed to be compactly generated locally compact, namely there exists a compact subset S_G of G such that $G = \bigcup_{n \geqslant 0} S_G^n$, and similarly for H. Given a compactly generated locally compact group G, we can define the word length by

$$|g|_G = \min\left\{n \geqslant 0 \mid g \in S_G^n\right\}$$

for every $g \in G$, which gives rise to the left-invariant word metric $(g, g') \mapsto |g^{-1}g'|_G$. Given $g \in G$ and $n \geqslant 0$, $B_G(g, n)$ will refer to the closed ball centered at g and of radius n. Denoting by λ_G the Haar measure of G, the *volume growth* of G is the map $V_G\colon \mathbb{N} \to \mathbb{N}$ defined by

$$V_G(n) \coloneqq \lambda_G\left(B_G(1_G, n)\right) = \lambda_G\left(\{g \in G \mid |g|_G \leqslant n\}\right)$$

for every integer $n \geqslant 0$.

Given non-decreasing maps $\varphi, \psi\colon \mathbb{R}_+ \to \mathbb{R}_+$, we say that ψ *dominates* φ, denoted $\varphi(t) \preccurlyeq \psi(t)$, if there exists a constant $C > 0$ such that $\varphi(t) \leqslant C\psi(Ct)$ for sufficiently large real numbers x. The maps φ and ψ are said to be *asymptotically equivalent*, denoted $\varphi(t) \simeq \psi(t)$, if $\varphi(t) \preccurlyeq \psi(t)$ and $\psi(t) \preccurlyeq \varphi(t)$, and the *asymptotic behaviour* refers to the class modulo $\simeq$. We also use the following stronger notions of domination:

- $\varphi(t) = O(\psi(t))$ means that there exists a constant $C > 0$ such that $\varphi(t) \leqslant C\psi(t)$ for sufficiently large real numbers t;
- $\varphi(t) = o(\psi(t))$ means that for every $\varepsilon > 0$, there exists $t_\varepsilon > 0$ such that $\varphi(t) \leqslant \varepsilon\psi(t)$ for every $t \geqslant t_\varepsilon$.

It is well known that two compact generating sets S_G and S'_G of a compactly generated locally compact group G provide word metrics which are bilipschitz equivalent [CDLH16, Proposition 4.B.4.(3)]. As a consequence, the volume growths associated to S_G and S'_G are asymptotically equivalent (the same holds for the notions of isoperimetric profile introduced below). Thus the reason why we forget the dependency on S_G in the notations $|.|_G$ and V_G is that we only care about the asymptotics of such quantities.

2.2 Quantitative measure equivalence of locally compact groups

The definition of measure equivalence in the locally compact setting is the following.

Definition 2.1. Let G and H be unimodular lcsc groups, with Haar measures λ_G and λ_H respectively. We say that G and H are *measure equivalent* if there exist a measure space (Ω, μ) and finite measure spaces (X_G, ν_G) and (X_H, ν_H), together with measured isomorphisms

$$i_G\colon (G \times X_G, \lambda_G \otimes \nu_G) \to (\Omega, \mu),$$
$$i_H\colon (H \times X_H, \lambda_H \otimes \nu_H) \to (\Omega, \mu),$$

such that the G- and H-actions defined by

$$g * i_G(g', x_G) = i_G(gg', x_G),$$
$$h * i_H(h', x_H) = i_H(hh', x_H)$$

commute.

Without loss of generality, we will consider X_G and X_H as subsets of Ω, so that they are fundamental domains of the G- and H-actions on (Ω, μ) (which are essentially free and preserve the measure), and i_G and i_H will be the maps defined by $i_G(g, x_G) = g * x_G$ and $i_H(h, x_H) = h * x_H$.

The quadruple (Ω, X_G, X_H, μ) is called a *measure equivalence coupling* between G and H. It induces actions on the fundamental domains, giving rise to cocycles.

Definition 2.2. A measure equivalence coupling (Ω, X_G, X_H, μ) between unimodular lcsc groups G and H induces a finite measure-preserving G-action on (X_H, ν_H) in the following way: for every $g \in G$ and every $x_H \in X_H$, $g \cdot x_H \in X_H$ is defined by the identity

$$(H * g * x_H) \cap X_H = \{g \cdot x_H\},$$

it is unique since X_H is a fundamental domain for the H-action on Ω.

This also yields a *cocycle* $\alpha\colon G \times X_H \to H$ uniquely defined by

$$\alpha(g, x_H) * g * x_H = g \cdot x_H,$$

or equivalently $\alpha(g, x_H) * g * x_H \in X_H$, for almost every $x_H \in X_H$ and every $g \in G$. We similarly define a finite measure-preserving H-action on (X_G, ν_G) and the associated cocycle $\beta\colon H \times X_G \to G$.

Remark 2.3. The cocycle $\alpha\colon G \times X_H \to H$ satisfies the cocycle identity

$$\forall g_1, g_2 \in G,\ \forall x_H \in X_H,\ \alpha(g_1 g_2, x_H) = \alpha(g_1, g_2 \cdot x_H)\alpha(g_2, x_H),$$

as well as $\beta\colon H \times X_G \to G$.

We now introduce the quantitative versions of measure equivalence. We first define the restrictions that we add on the cocycles.

Definition 2.4. Let G and H be two unimodular locally compact compactly generated groups. Let $G \curvearrowright (X, \nu)$ be a measure-preserving action on a finite measure space and let $c\colon G \times X \to H$ be an H-valued cocycle (i.e. a H-valued map satisfying the cocycle identity). Given a non-decreasing map $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that $\varphi(Ct) = O(\varphi(t))$ for every constant $C > 0$, we say that the cocycle $c\colon G \times X \to H$ is *φ-integrable* if

$$\sup_{g \in S_G} \int_X \varphi(|c(g, x)|_H) \mathrm{d}\nu(x) < +\infty$$

Examples of non-decreasing maps $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\varphi(Ct) = O(\varphi(t))$ for every constant $C > 0$ are:

- non-decreasing subadditive maps, for instance $\varphi(t) = t^p$ with $p \in]0, 1]$;
- non-decreasing maps φ such that $t \mapsto t/\varphi(t)$ is non-decreasing, for instance $\varphi(t) = \log(1 + t)$. Indeed, if $C \leqslant 1$, then we immediately get $\varphi(Ct) \leqslant \varphi(t)$, and if $C > 1$, then we have $C/\varphi(Ct) \geqslant t/\varphi(t)$, namely $\varphi(Ct) \leqslant C\varphi(t)$.

Note that this definition does not depend on the choice of the compact generating set for H, by assumption on φ, whereas it seems to depend on the compact generating set S_G for G. Two remarks are in order.

- First, it is possible to replace S_G by S_G^{-1}. This is a consequence of the formula $c(g, x) = c(g^{-1}, g \cdot x)^{-1}$ (provided by the cocycle identity) and the invariance of the measure.

- Secondly, if φ is subadditive, being φ-integrable is the same as saying that $\int_X \varphi(|c(g,x)|_H)\mathrm{d}\nu(x)$ is finite for all g in G without any uniform condition on the bound, by the same argument as in [BFS13, Appendix A.2] (see also the proof of Proposition 6.3 which yields the main ingredients). For instance, this is the case for $\varphi(t) = t^p$ with $0 < p \leqslant 1$. In particular, this implies that the notion of φ-integrability does not depend on S_G for such maps φ.

Let us now define the quantitative versions of measure equivalence.

Definition 2.5. Let G and H be two unimodular locally compact compactly generated groups. Let $\varphi, \psi \colon \mathbb{R}_+ \to \mathbb{R}_+$ be non-decreasing maps satisfying $\varphi(Ct) = O(\varphi(t))$ and $\psi(Ct) = O(\psi(t))$ for every constant $C > 0$. We say that (Ω, X_G, X_H, μ) is a (φ,ψ)-*integrable measure equivalence coupling* from G to H if it is a measure equivalence coupling between G and H such that the associated cocycles $\alpha\colon G \times X_H \to H$ and $\beta\colon H \times X_G \to G$ satisfy the following: α is φ-integrable and β is ψ-integrable.

For $p > 0$, we write L^p instead of φ or ψ if we consider the map $t \mapsto t^p$, and we write L^0 when no requirement is made on the corresponding cocycle. Finally, a measure equivalence coupling is φ-*integrable* if it is (φ,φ)-integrable.

2.3 Isoperimetric profile of locally compact groups

The goal of this subsection is to collect the relevant properties of the isoperimetric profile for locally compact groups. To this end, the crucial point is the definition of the L^p-*norm* of the gradient. Recall that for a countable group Γ admitting a finite generating set S_Γ, it is defined as

$$\|\nabla_\Gamma f\|_p^p = \sum_{s\in S_\Gamma}\sum_{\gamma\in\Gamma} |f(\gamma) - f(s^{-1}\gamma)|^p = \sum_{s\in S_\Gamma} \|f - \lambda(s)f\|_p^p$$

for every $f\colon \Gamma \to \mathbb{R}$ of finite support, where $\lambda\colon \Gamma \curvearrowright \mathrm{L}^p(\Gamma)$ is the left-regular representation. Given G a locally compact group with compact generating subset S_G, and with fixed Haar measure λ_G, the most natural generalization is

$$\|\nabla_G^{\mathrm{int}} f\|_p^p = \int_{s\in S_G} \|f - \lambda(s)f\|_p^p \,\mathrm{d}\lambda_G(s)$$

for $f \in \mathrm{L}^p(G)$ with support of finite measure. We have another possible definition of the L^p-norm of the gradient, given by

$$\|\nabla_G^{\sup} f\|_p = \sup_{s\in S_G} \|f - \lambda(s)f\|_p.$$

This second definition appears in our calculations when bounding above some quantities (see the proof of Lemma 5.7), so we want it to be equivalent to the first one for the underlying isoperimetric profiles to be asymptotically equivalent. Note that we easily get $\|\nabla_G^{\mathrm{int}} f\|_p^p \leqslant \lambda(S_G)\|\nabla_G^{\sup} f\|_p^p$.

The comparison between these notions has already been done in [Tes08, Propositions 7.1 and 7.2]. Here we provide another proof.

Proposition 2.6. *There exists a constant $C > 0$ such that*

$$\frac{1}{C}\|\nabla_G^{\sup} f\|_p \leqslant \|\nabla_G^{\text{int}} f\|_p \leqslant C\|\nabla_G^{\sup} f\|_p$$

for every $f \in \mathrm{L}^p(G)$.

Proof. Let us consider $Z^1(G, \lambda)$, the set of cocycles for the left-regular representation on $\mathrm{L}^p(G)$, namely the set of maps $b\colon G \to \mathrm{L}^p(G)$ satisfying the 1-cocycle relation

$$b(gh) = b(g) + \lambda(g)b(h)$$

for every $g, h \in G$. By [LNP25, Proposition 1.13], the followings are equivalent norms on $Z^1(G, \lambda)$:

$$\|b\|_{\sup} := \sup_{s \in S_G} \|b(s)\|_p,$$

$$\|b\|_{\text{int}} := \left(\int_{s \in S_G} \|b(s)\|_p^p \,\mathrm{d}\lambda(s)\right)^{\frac{1}{p}}.$$

The result then follows from the fact that, for every $f \in \mathrm{L}^p(G)$, we have $\|\nabla_G^{\text{int}} f\|_p = \|b\|_{\text{int}}$ and $\|\nabla_G^{\sup} f\|_p = \|b\|_{\sup}$ with $b\colon G \to L^p(G)$ given by $b(g) = f - \lambda(g)f$. Note that b lies in $Z^1(G, \lambda)$ since it is a coboundary. □

From this, we deduce that the underlying isoperimetric profiles are asymptotically equivalent. Let us fix the notations in the following.

Definition 2.7. Let G be a compactly generated locally compact group, with a compact generating set S_G. Let $p \geqslant 1$. Then we define the following two notions of L^p-*isoperimetric profiles*:

$$j_{p,G}^{\sup}(v) = \sup_{\substack{f \in \mathrm{L}^p(G),\\ \lambda_G(\mathrm{supp}(f)) \leqslant v}} \frac{\|f\|_p}{\|\nabla_G^{\sup} f\|_p},$$

$$j_{p,G}^{\text{int}}(v) = \sup_{\substack{f \in \mathrm{L}^p(G),\\ \lambda_G(\mathrm{supp}(f)) \leqslant v}} \frac{\|f\|_p}{\|\nabla_G^{\text{int}} f\|_p}.$$

Since these two maps are asymptotically equivalent, we will denote by $j_{p,G}$ their asymptotic behaviour.

Two compact generating sets give rise to asymptotically equivalent isoperimetric profiles. Finally, note that if G is unimodular, this is also the same as defining the isoperimetric profile with respect to the right-regular representation.

3 A preparatory lemma

Given a measure equivalence coupling (Ω, X_G, X_H, μ) between locally compact groups G and H, and given the induced actions $G \curvearrowright (X_H, \nu_H)$ and $H \curvearrowright (X_G, \nu_G)$ of the groups on the fundamental domains, we set

$$R_Y^G(x) \coloneqq \{g \in G \mid g \cdot x \in Y\}$$

for every $Y \subset X_H$ and every $x \in X_H$, and similarly $R_Y^H(x)$ for every $Y \subset X_G$ and $x \in X_G$.

The goal of this section is to prove the following key lemma which will be useful for our main theorems.

Lemma 3.1. *Let $\varphi, \psi \colon \mathbb{R}_+ \to \mathbb{R}_+$ be non-decreasing maps and let G and H be unimodular non-discrete locally compact groups. If there exists a (φ, ψ)-integrable measure equivalence coupling from G to H, then there exists a measure equivalence coupling (Ω, X_G, X_H, μ) satisfying the followings:*

(P1) there exists a constant $C > 0$ such that $\nu_G|_{X_G \cap X_H} = C \cdot \nu_H|_{X_G \cap X_H}$ and $\nu_G(X_G \cap X_H) = C \cdot \nu_H(X_G \cap X_H) > 0$;

(P2) the associated cocycles $\alpha \colon G \times X_H \to H$ and $\beta \colon H \times X_G \to G$ are respectively φ- and ψ-integrable (in particular the new measure equivalence coupling is also (φ, ψ)-integrable from G to H);

(P3) the map

$$\alpha(., x) \colon (R^G_{X_G \cap X_H}(x), \lambda_G) \to (\alpha(R^G_{X_G \cap X_H}(x), x), \lambda_H)$$

is measure preserving for almost every $x \in X_H$, as well as the map

$$\beta(., x) \colon (R^H_{X_G \cap X_H}(x), \lambda_H) \to (\beta(R^H_{X_G \cap X_H}(x), x), \lambda_G)$$

for almost every $x \in X_G$.

If Property *(P3)* is required only for the points x in the intersection $X_G \cap X_H$, this lemma is relatively straightforward in the case of countable groups. Indeed, up to some translation, the fundamental domains intersect non-trivially, and it is easy to verify that $\alpha(., x) \colon R^G_{X_G \cap X_H}(x) \to \alpha(R^G_{X_G \cap X_H}(x), x)$ and $\beta(., x) \colon R^H_{X_G \cap X_H}(x) \to \beta(R^H_{X_G \cap X_H}(x), x)$ are bijective if x lies in the intersection $X_G \cap X_H$. Since the Haar measures are the counting measures in this case, we immediately get that these bijections are measure preserving. In the non-discrete case, it is much harder and the techniques in [DLIT25, Proposition A.1] will help us, it consists in using cross-sections to slightly modify the fundamental domains so that Property *(P3)* in the lemma hold for every x in the intersection $X_G \cap X_H$.

To get Property *(P3)* for almost every $x \in X_H$ for α, and $x \in X_G$ for β, and not only for x in the intersection, we require the induced actions to be ergodic, ensuring that almost every orbit in X_G (resp. in X_H) visits the intersection $X_G \cap X_H$. To this end, we follow the proof of [KKR21a, Proposition 2.17 (ii)].

Finally, we must verify that (φ, ψ)-integrability is preserved under the transformations we apply. First, for the transformations used in [KKR21a, Proposition 2.17 (ii)], preservation follows from the ergodic decomposition theorem. Secondly, in [DLIT25, Proposition A.1], the new fundamental domains are obtained by translating the previous ones by elements of a compact subset of $G \times H$, ensuring a uniform bound on the difference between the length norms of the new and original cocycles.

Proof of lemma 3.1. Let (Ω, X_G, X_H, μ) be a (φ, ψ)-integrable measure equivalence coupling from G to H. As in the definition, we denote by i_G, i_H the measured isomorphisms

$$i_G \colon (G \times X_G, \lambda_G \otimes \nu_G) \to (\Omega, \mu),$$

$$i_H \colon (H \times X_H, \lambda_H \otimes \nu_H) \to (\Omega, \mu),$$

where ν_G and ν_H are the finite measures on X_G and X_H.

Step 1: assuming that μ is ergodic. Let us first follow the proof of [KKR21a, Proposition 2.17 (ii)]. We have to be careful with the notations, since our notations $X_H, X_G, \mu, \nu_G, \nu_H, i_G, i_H$ refer to the notations $X, Y, \eta, \mu, \nu, i, j$ in [KKR21a]. By the Ergodic Decomposition Theorem, there is a standard probability space (Z, ζ) and a familly $((\nu_H)_z)_{z \in Z}$ of ergodic measures for the induced action $G \curvearrowright X_H$, such that

$$\nu_H(A) = \int_Z (\nu_H)_z(A) \, \mathrm{d}\zeta(z) \tag{7}$$

for every measurable subset $A \subset X_H$. We then set $\mu_z \coloneqq (i_H)_\star[\lambda_H \otimes (\nu_H)_z]$, this is a measure on Ω. It is shown in the proof of [KKR21a, Proposition 2.17 (ii)] that the measures μ_z are σ-finite and ergodic (with respect to the $(G \times H)$-action on Ω) for almost all $z \in Z$, and give rise to measures $(\nu_G)_z$ on X_G such that

- for every $z \in Z$, $\lambda_G \otimes (\nu_G)_z = (i_G^{-1})_\star \mu_z$;
- for almost every $z \in Z$, $(\nu_G)_z$ is a finite measure and is ergodic for the induced action $H \curvearrowright X_G$.

Given z in a conull subset of Z (a subset on which the above properties hold), we now consider the measure equivalence coupling $(\Omega, \mu_z, X_G, X_H)$, with the finite measures $(\nu_G)_z$ and $(\nu_H)_z$ on X_G and X_H. By (7), we have

$$\int_{X_H} \varphi(|\alpha(g,x)|_H) \, \mathrm{d}\nu_H = \int_Z \left(\int_{X_H} \varphi(|\alpha(g,x)|_H) \, \mathrm{d}(\nu_H)_z(x) \right) \, \mathrm{d}\zeta(z),$$

so the cocycle α is φ-integrable with respect to $(\nu_H)_z$ for almost every $z \in Z$. We can also find a desintegration for ν_G, similarly to (7), and prove that the other cocycle β is ψ-integrable with respect to $(\nu_G)_z$ for almost every $z \in Z$. So the couplings $(\Omega, \mu_z, X_G, X_H)$ are (φ, ψ)-integrable for almost all $z \in Z$.

Step 2: Property *(P3)* for x in the intersection $X_G \cap X_H$. Let us fix z in a conull subset of Z for which the desired properties hold. By [DLIT25, Proposition A.1], we can choose new fundamental domains $X'_{G,z}$ and $X'_{H,z}$ satisfying Property *(P1)*, and so that the new cocycles α'_z and β'_z satisfy Property *(P3)* for every $x \in X'_{G,z} \cap X'_{H,z}$. Furthermore, these cocycles are φ- and ψ-integrable, since their norms are bounded above by the norms of the previous cocycles with an additive term (see the proof of [DLIT25, Proposition A.1]).

Step 3: ergodicity for the induced actions. For every z in a conull subset of Z, we have built a measure equivalence coupling $(\Omega, \mu_z, X'_{G,z}, X'_{H,z})$ where μ_z is ergodic and the associated cocycles satisfy the following: for almost every $y \in X'_{G,z} \cap X'_{H,z}$, the maps

$$\alpha'_z(.,y)\colon (R^G_{X'_{G,z}\cap X'_{H,z}}(y), \lambda_G) \to (\alpha'_z(R^G_{X'_{G,z}\cap X'_{H,z}}(y), y), \lambda_H) \tag{8}$$

and

$$\beta'_z(.,y)\colon (R^H_{X'_{G,z}\cap X'_{H,z}}(y), \lambda_H) \to (\beta'_z(R^H_{X'_{G,z}\cap X'_{H,z}}(y), y), \lambda_G) \tag{9}$$

are measure preserving, and the goal is now to extend it to $y \in X_H$ (for α'_z) and to $y \in X_G$ (for β'_z), using ergodicity. Coming back to Step 1 of this proof, or equivalently to the proof of [KKR21a, Proposition 2.17 (ii)], we know that the ergodic measure μ_z gives rise to ergodic finite measures $(\nu'_H)_z$ and $(\nu'_G)_z$ for the induced actions $G \curvearrowright X'_{H,z}$ and $H \curvearrowright X'_{G,z}$. To conclude, for almost every $x \in X'_{H,z}$, the set $R^G_{X'_{G,z}\cap X'_{H,z}}(x)$ is not empty and we pick a point g_x. Then for every $g \in R^G_{X'_{G,z}\cap X'_{H,z}}(x)$, the cocycle identity gives

$$\alpha'_z(g,x) = \alpha(gg_x^{-1}, g_x \cdot x)\alpha(g_x, x)$$

where $g_x \cdot x$ lies in the intersection $X'_{G,z} \cap X'_{H,z}$, so it follows from unimodularity and from the fact that the map (8) (with $y = g_x \cdot x$) is measure preserving, that

$$\alpha'_z(.,x)\colon (R^G_{X'_{G,z}\cap X'_{H,z}}(x), \lambda_G) \to (\alpha'_z(R^G_{X'_{G,z}\cap X'_{H,z}}(x), x), \lambda_H)$$

is measure preserving. The same holds for β'_z. This completes the proof. □

4 Behaviour of volume growth

In this section, we prove Theorem A that we now recall.

Theorem 4.1. *Let G and H be non-discrete unimodular locally compact compactly generated groups and $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a non-decreasing and subadditive map. If there exists a (φ, L^0)-integrable measure equivalence coupling from G to H, then*

$$V_G(n) \preccurlyeq V_H(\varphi^{-1}(n)).$$

We begin with a simple result, found for instance in [Aus16, Proof of Lemma B.11].

Lemma 4.2. *Let $G \curvearrowright (X,\nu)$ be a pmp action of a locally compact group on a probability space. Let Y be a measurable subset of X, of positive measure. Then we have*

$$\int_X \frac{\lambda_G(R^G_Y(x) \cap B_G(1_G, n))}{V_G(n)}\, d\nu(x) = \nu(Y).$$

Proof. The proof follows directly from the invariance of the Haar measure:

$$\begin{aligned}
&\int_X \lambda_G(R_Y^G(x) \cap B_G(1_G, n))\; d\nu(x) \\
=&\int_X \int_{B_G(1_G,n)} \mathbb{1}_{x\in g^{-1}Y}\; d\lambda_G(g)\; d\nu(x) \\
=&\int_{B_G(1_G,n)} \int_X \mathbb{1}_{x\in g^{-1}Y}\; d\nu(x)\; d\lambda_G(g) \\
=&\int_{B_G(1_G,n)} \nu(g^{-1}Y)\; d\lambda_G(g) \\
=&\int_{B_G(1_G,n)} \nu(Y)\; d\lambda_G(g) \\
=&\lambda_G(B_G(1_G,n))\nu(Y) \\
=&V_G(n)\nu(Y),
\end{aligned}$$

which completes the result. □

Proof of Theorem 4.1. Without loss of generality, we assume that the properties given by Lemma 3.1 hold. Let us also assume that $\nu_H(X_H) = 1$. Let $Y := X_G \cap X_H$. Let $K, K' > 0$ be positive constants such that

$$\mu(Y) + \left(1 - \frac{1}{K}\right)\left(1 - \frac{1}{K'}\right) - 1 > 0.$$

By [DKLMT22, Lemma 2.24] which also holds in our framework, there exists a constant $C > 0$ such that for every $g \in G$,

$$\int_{X_H} \varphi(|\alpha(g,x)|_H) d\nu_H(x) \leqslant C|g|_G.$$

We set

$$X_1 := \left\{ x \in X_H \;:\; \int_{B_G(1_G,n)} \frac{\varphi(|\alpha(g,x)|_H)}{|g|_G} d\lambda_G(g) \leqslant CKV_G(n) \right\}$$

and for every $x \in X_H$,

$$G_x := \{g \in B_G(1_G, n) \mid \varphi(|\alpha(g,x)|_H) \leqslant CKK'|g|_G\}.$$

By Markov's inequality on the probability space (X_H, ν_H), we have

$$\nu_H(X_1) \geqslant 1 - \frac{1}{K}. \tag{10}$$

Then Markov's inequality on the probability space $\left(B_G(1_G, n), \frac{\lambda_G}{V_G(n)}\right)$ gives for every $x \in X_1$,

$$\lambda_G(G_x) \geqslant \left(1 - \frac{1}{K'}\right) V_G(n). \tag{11}$$

To prove the theorem, we now derive bounds of $I \coloneqq \int_{X_H} \lambda_G(R_Y(x) \cap G_x) d\nu_H(x)$.

On the one hand, using the inclusion

$$R_Y(x) \cap B_G(1_G, n) \subset (R_Y(x) \cap G_x) \cup (B_G(1_G, n) \backslash G_x)$$

for every $x \in X_H$, we have

$$\begin{aligned} I &\geqslant \int_{X_H} \left(\lambda_G(R_Y(x) \cap B_G(1_G, n)) + \lambda_G(G_x) - \lambda_G(B_G(1_G, n))\right) d\nu_H(x) \\ &\geqslant \int_{X_H} \lambda_G(R_Y(x) \cap B_G(1_G, n)) d\nu_H(x) + \int_{X_1} \lambda_G(G_x) d\nu_H(x) - \int_{X_H} V_G(n) d\nu_H(x) \\ &\geqslant V_G(n) \underbrace{\left(\nu_H(Y) + \left(1 - \frac{1}{K}\right)\left(1 - \frac{1}{K'}\right) - 1\right)}_{=: K'' > 0}, \end{aligned}$$

where we also apply Lemma 4.2 and Equations (10) and (11).

On the other hand, using the definition of G_x and Property *(P3)* of Lemma 3.1, we get

$$\begin{aligned} I &\leqslant \int_{X_H} \int_{R_Y^G(x)} \mathbb{1}_{\alpha(g,x) \in B_H(1_H, \varphi^{-1}(CKK'n))} \, d\lambda_G(g) \, d\nu_H(x) \\ &= \int_{X_H} \int_{\alpha(R_Y^G(x), x)} \mathbb{1}_{h \in B_H(1_H, \varphi^{-1}(CKK'n))} \, d\lambda_H(h) \, d\nu_H(x) \\ &\leqslant V_H(\varphi^{-1}(CKK'n)). \end{aligned}$$

Hence $K'' V_G(n) \leqslant V_H(\varphi^{-1}(CKK'n))$, completing the proof. □

5 Behaviour of isoperimetric profiles

The goal of this section is to prove Theorem B and C, that we now recall.

Theorem 5.1. *Let G and H be non-discrete unimodular locally compact compactly generated groups and $p \geqslant 1$. If there exists an $(\mathrm{L}^p, \mathrm{L}^0)$-integrable measure equivalence coupling from G to H, then*

$$j_{p,H}(n) \preccurlyeq j_{p,G}(n).$$

Theorem 5.2. *Let G and H be non-discrete unimodular locally compact compactly generated groups and $\varphi \colon \mathbb{R}_+ \to \mathbb{R}_+$ be a non-decreasing map such that $t \mapsto t/\varphi(t)$ is non-decreasing. If there exists a (φ, L^0)-integrable measure equivalence coupling from G to H, then*

$$\varphi \circ j_{1,H}(n) \preccurlyeq j_{1,G}(n).$$

5.1 Outlines of the proofs

As for Theorem A, we assume without loss of generality that the properties given by Lemma 3.1 hold.

To prove Theorems 5.1 and 5.2, we fix a map $f\colon H \to \mathbb{R}$ whose support has finite measure. Given an integer $v \geqslant 0$, f will be viewed as a map almost realising $j^{\sup}_{1,H}(v)$, namely $\lambda_H(\mathrm{supp} f) \leqslant v$ and $\frac{\|f\|_p}{\|\nabla^{\sup}_H f\|_p}$ is almost equal to $j_{1,H}(v)$.

From this map f and the measure equivalence coupling, we deduce, via the identification $\Omega \simeq X_H \times H$, a new map $\tilde{f}\colon \Omega \to \mathbb{R}$ defined by

$$\tilde{f}(h * x) = f(h^{-1})$$

for every $h \in H$ and $x \in X_H$. Moreover, from the identification $\Omega \simeq X_G \times G$, we get for every $x \in X_G$ a map $f_x\colon G \to \mathbb{R}$ defined by

$$f_x(g) = \tilde{f}(g * x)$$

for every $g \in G$.

Given $x \in X_G$, there exists another formula for the definition of f_x. Indeed, by definition, given $g \in G$, $f_x(g)$ is equal to $f(h^{-1})$ for some $h \in H$ satisfying $g * x \in h * X_H$. There exists a unique $h_x \in H$ such that $h_x * x \in X_H$, so $h^{-1} * g * x$ is a point in X_H which is also equal to $h^{-1}h_x^{-1} * g * h_x * x$, meaning that $h^{-1}h_x^{-1} = \alpha(g, h_x * x)$. To conclude, f_x is defined by

$$f_x(g) = f(\alpha(g, h_x * x)h_x) \tag{12}$$

for every $g \in G$.

To sum up, a map $f\colon H \to \mathbb{R}$ with support of finite measure gives rise to a random map $f_x\colon G \to \mathbb{R}$ (random in $x \in X_G$ where X_G is endowed with the finite measure ν_G), given by formula (12). Furthermore, f and the maps f_x are also related by a map $\tilde{f}\colon \Omega \to \mathbb{R}$ coming from the measure equivalence coupling.

Step 1. By Proposition 5.3, the mean value of $\lambda_G(\mathrm{supp} f_x)$ is bounded above by $\lambda_H(\mathrm{supp} f)$ with some multiplicative constant.

Step 2. Proposition 5.4 is about the L^p-norm of f_x: its mean value is bounded below by $\|f\|_p$.

Step 3. Propositions 5.8 and 5.9 finally deal with the mean of $\|\nabla^{\mathrm{int}}_G f_x\|_p^p$ and give an upper bound with $\|\nabla^{\sup}_H f\|_p^p$. These propositions are decomposed in three lemmas which use the function $\tilde{f}$, while the statements in the previous steps only use the formula (12). Lemma 5.5 states that the mean value of $\|\nabla^{\mathrm{int}}_G f_x\|_p^p$ is exactly

$$\|\nabla^{\mathrm{int}}_G \tilde{f}\|_p^p \coloneqq \int_{S_G} \int_\Omega \left|\tilde{f}(s^{-1} * \omega) - \tilde{f}(\omega)\right|^p \mathrm{d}\mu(\omega)\ \mathrm{d}\lambda_G(s),$$

and Lemmas 5.6 and 5.7 provides an upper bound of $\|\nabla^{\mathrm{int}}_G \tilde{f}\|_p^p$ given by $\|\nabla^{\sup}_H f\|_p^p$.

Theorems 5.1 and 5.2 then follow from a pigeonhole principle.

5.2 Proof of the theorems

We follow the steps described in Section 5.1 to prove Theorems 5.1 and 5.2. In the following propositions and lemmas, we assume that the properties given by Lemma 3.1 hold, and we fix a map $f\colon H \to \mathbb{R}$ giving rise to a random map $f_x\colon G \to \mathbb{R}$ as described above. Moreover G and H are always assumed to be non-discrete unimodular locally compact compactly generated groups.

Proposition 5.3. *With the same conventions and notations as in Section 5.1, there exists a positive constant K (independent of f) such that*

$$\int_{X_G} \lambda_G(\mathrm{supp} f_x)\ d\nu_G(x) \leqslant K \lambda_H(\mathrm{supp} f).$$

Proof. Let $x \in X_G$. Given $g \in G$ such that $f_x(g) \neq 0$, the relation between f and f_x (see the paragraph before the formula (12)) implies that there exists $h \in \mathrm{supp} f$ such that $h * g * x \in X_H$, so $x \in g^{-1} * (\mathrm{supp} f)^{-1} * X_H$. We thus get

$$\begin{aligned}\int_{X_G} \lambda_G(\mathrm{supp} f_x)\ d\nu_G(x) &= \int_{X_G}\int_G \mathbb{1}_{f_x(g)\neq 0}\ d\lambda_G(g)\ d\nu_G(x)\\ &\leqslant \int_{X_G}\int_G \mathbb{1}_{x\in g^{-1}*(\mathrm{supp} f)^{-1}*X_H}\ d\lambda_G(g)\ d\nu_G(x)\\ &= \int_G \nu_G(X_G \cap (g^{-1} * (\mathrm{supp} f)^{-1} * X_H))\ d\lambda_G(g)\\ &= \mu((\mathrm{supp} f)^{-1} * X_H)\\ &= \lambda_H((\mathrm{supp} f)^{-1})\ \nu_H(X_H)\\ &= \lambda_H(\mathrm{supp} f)\ \nu_H(X_H),\end{aligned}$$

where the last equality uses unimodularity. □

Proposition 5.4. *With the same conventions and notations as in Section 5.1, there exists a positive constant K' (independent of f) such that for every $p \geqslant 1$,*

$$\int_{X_G} \|f_x\|_p^p\ d\nu_G(x) \geqslant K' \|f\|_p^p$$

Proof. Let $x \in X_G$ and $g \in G$. Then we have $h * g * x \in X_H$ with $h = \alpha(g, h_x * x)h_x$. Let us denote $Y \coloneqq X_G \cap X_H$ and let us more particularly assume that $h * g * x$ lies in Y. On the one hand, the definition of the induced action $H \curvearrowright (X_G, \nu_G)$ implies $h * g * x = h \cdot x$. On the other hand, we get from the definition of the induced action $G \curvearrowright (X_H, \nu_H)$ that $h * g * x = g \cdot (h_x * x)$. We thus get $g \in R_Y^G(h_x * x) \iff h \in R_Y^H(x)$, in other words

$$\left\{\alpha\left(g, h_x * x\right) h_x \mid g \in R_Y^G(h_x * x)\right\} = R_Y^H(x).$$

Using the fact that the surjective map

$$\alpha(., h_x * x)h_x\colon (R_Y^G(h_x * x), \lambda_G) \to (R_Y^H(x), \lambda_H)$$

is measure preserving (by the properties of Lemma 3.1), we thus get

$$\begin{aligned}
\int_{X_G} \|f_x\|_p^p \, d\nu_G(x) &= \int_{X_G} \int_G |f(\alpha(g, h_x * x)h_x)|^p \, d\lambda_G(g) \, d\nu_G(x) \\
&\geqslant \int_{X_G} \int_{R_Y^G(h_x * x)} |f(\alpha(g, h_x * x)h_x)|^p \, d\lambda_G(g) \, d\nu_G(x) \\
&= \int_{X_G} \int_{R_Y^H(x)} |f(h)|^p \, d\lambda_H(h) \, d\nu_G(x) \\
&= \int_{X_G} \int_H |f(h)|^p \mathbb{1}_{x \in h^{-1} \cdot Y} \, d\lambda_H(h) \, d\nu_G(x) \\
&= \int_H |f(h)|^p \nu_G(h^{-1} \cdot Y) \, d\lambda_H(h) \\
&= \nu_G(Y) \|f\|_p^p
\end{aligned}$$

and we are done. □

We now use $\tilde{f} \colon \Omega \to \mathbb{R}$ which relates f and the maps f_x (for $x \in X_G$) via the measure equivalence coupling. Recall that the quantity $\|\nabla_G^{\text{int}} \tilde{f}\|_p$ is defined by

$$\|\nabla_G^{\text{int}} \tilde{f}\|_p^p \coloneqq \int_{S_G} \int_\Omega \left| \tilde{f}(s^{-1} * \omega) - \tilde{f}(\omega) \right|^p \mathrm{d}\mu(\omega) \, \mathrm{d}\lambda_G(s).$$

Lemma 5.5. *With the same conventions and notations as in Section 5.1, we have for every* $p \geqslant 1$,

$$\int_{X_G} \|\nabla_G^{\text{int}} f_x\|_p^p \, d\nu_G(x) = \|\nabla_G^{\text{int}} \tilde{f}\|_p^p$$

Proof. It is simply a change of variable $(g, x) \mapsto g * x$ from $(G \times X_G, \lambda_G \otimes \nu_G)$ to (Ω, μ). □

Lemma 5.6. *With the same conventions and notations as in Section 5.1, we have for every* $p \geqslant 1$,

$$\|\nabla_G^{\text{int}} \tilde{f}\|_p^p = \int_{S_G} \int_{X_H} \|\lambda(\alpha(s^{-1}, x)^{-1}) f - f\|_p^p \, d\nu_H(x) \, d\lambda_G(s)$$

Proof. The change of variable $h * x \mapsto (h, x)$ from (Ω, μ) to $(H \times X_H, \lambda_H \otimes \nu_H)$ gives

$$\|\nabla_G^{\text{int}} \tilde{f}\|_p^p = \int_{S_G} \int_{H \times X_H} |\tilde{f}(s^{-1} * h * x) - \tilde{f}(h * x)|^p \, d(\lambda_H \otimes \nu_H)(h, x) \, d\lambda_G(s).$$

Given $s \in S_G$ and $x \in X_H$, we have $s^{-1} * x = \alpha(s^{-1}, x)^{-1} * (s^{-1} \cdot x)$. Now using the definition of $\tilde{f}$ from f, we obtain

$$\tilde{f}(s^{-1} * h * x) = \tilde{f}(h\alpha(s^{-1}, x)^{-1} * (s \cdot x)) = f(\alpha(s^{-1}, x)h^{-1})$$

and

$$\tilde{f}(h * x) = f(h^{-1})$$

since x and $s^{-1} \cdot x$ lie in X_H. Unimodularity of H now implies that $h \mapsto h^{-1}$ preserves the Haar measure λ_H and we are done. □

Lemma 5.7. *With the same conventions and notations as in Section 5.1, we have for any $p \geqslant 1$ and any h in H,*

$$||\lambda(h^{-1})f - f||_p \leqslant |h|_H ||\nabla_H^{\sup} f||_p$$

Proof. By the definition of $|h|_H =: n$, we have that $h^{-1} = s_1 s_2 \dots s_n$ where each s_i belongs to $S_H \cup S_H^{-1}$ for $i \in \{1, \cdots, n\}$. Then it is clear that:

$$\begin{aligned} ||\lambda(h^{-1})f - f||_p &\leqslant \sum_{i=0}^{n-1} ||\lambda(s_1 \cdots s_{i+1})f - \lambda(s_1 \cdots s_i)f||_p \\ &= \sum_{i=0}^{n-1} ||\lambda(s_{i+1})f - f||_p \\ &\leqslant n||\nabla_H^{\text{int}} f||_p = |h|_H ||\nabla_H^{\text{int}} f||_p, \end{aligned}$$

concluding the proof of the lemma. □

Proposition 5.8. *Let us keep the same conventions and notations as in Section 5.1. Let $p \geqslant 1$. Assume that*

$$C_p := \int_{S_G} \int_{X_H} |\alpha(s^{-1}, x)|_H^p \; d\nu_H(x) \; d\lambda_G(s)$$

is finite, then we have

$$\int_{X_G} \|\nabla_G^{\text{int}} f_x\|_p^p \; d\nu_G(x) \leqslant C_p \|\nabla_H^{\sup} f\|_p^p.$$

Proof. By Lemmas 5.5 and 5.6, we have

$$\int_{X_G} \|\nabla_G^{\text{int}} f_x\|_p^p \; d\nu_G(x) = \int_{S_G} \int_{X_H} \|\lambda(\alpha(s^{-1}, x)^{-1})f - f\|_p^p \; d\nu_H(x) \; d\lambda_G(s).$$

Lemma 5.7 now implies

$$||\lambda(\alpha(s^{-1}, x)^{-1})f - f||_p^p \leqslant |\alpha(s^{-1}, x)|_H^p ||\nabla_H^{\sup} f||_p^p$$

for every $s \in S_G$ and $x \in X_G$, and the result follows. □

Proposition 5.9. *Let us keep the same conventions and notations as in Section 5.1. Let $\varphi \colon \mathbb{R}_+ \to \mathbb{R}_+$ be a non-decreasing map such that $t \mapsto t/\varphi(t)$ is non-decreasing. Assume that*

$$C_\varphi := \int_{S_G} \int_{X_H} \varphi(|\alpha(s^{-1}, x)|_H \|\nabla_H^{\sup} f\|_1) \; d\nu_H(x) \; d\lambda_G(s)$$

is finite, then we have

$$\int_{X_G} \|\nabla_G^{\text{int}} f_x\|_1 \; d\nu_G(x) \leqslant C_\varphi \frac{2\|f\|_1}{\varphi(2\|f\|_1)}.$$

Proof. Using again Lemmas 5.5 and 5.6, we have

$$\begin{aligned}
&\int_{X_G} \|\nabla_G^{\mathrm{int}} f_x\|_1 \; d\nu_G(x) \\
&= \int_{S_G}\int_{X_H} \|\lambda(\alpha(s^{-1},x)^{-1})f - f\|_1 \; d\nu_H(x)\; d\lambda_G(s) \\
&= \int_{S_G}\int_{X_H} \frac{\|\lambda(\alpha(s^{-1},x)^{-1})f - f\|_1}{\varphi(\|\lambda(\alpha(s^{-1},x)^{-1})f - f\|_1)}\varphi(\|\lambda(\alpha(s^{-1},x)^{-1})f - f\|_1)\; d\nu_H(x)\; d\lambda_G(s).
\end{aligned}$$

Let $(s,x) \in S_G \times X_H$. First, the monotonicity of $t \mapsto t/\varphi(t)$ and the inequality $\|\lambda(\alpha(s^{-1},x)^{-1})f - f\|_1 \leqslant 2\|f\|_1$ imply

$$\frac{\|\lambda(\alpha(s^{-1},x)^{-1})f - f\|_1}{\varphi(\|\lambda(\alpha(s^{-1},x)^{-1})f - f\|_1)} \leqslant \frac{2\|f\|_1}{\varphi(2\|f\|_1)}. \tag{13}$$

Secondly, the monotonicity of φ and Lemma 5.7 give

$$\varphi(\|\lambda(\alpha(s^{-1},x)^{-1})f - f\|_1) \leqslant \varphi(|\alpha(s^{-1},x)|_H \|\nabla_H^{\sup} f\|_1). \tag{14}$$

We finally deduce the result from Inequalities (13) and (14). □

Proof of Theorem 5.1. We assume without loss of generality that the measure equivalence coupling satisfies the properties provided by Lemma 3.1. Let v be a positive real number, let $f\colon H \to \mathbb{R}$ whose support has measure less than v and the associated maps f_x as explained in Section 5.1. Let K, K', C_p be the constants introduced in Propositions 5.3, 5.4 and 5.8. Note that C_p is finite since the cocycle is L^p and by the comments after Definition 2.4 (namely S_G can be replaced by S_G^{-1}).

We claim that the set

$$A \coloneqq \left\{ x \in X_G \;:\; \|\nabla_G^{\mathrm{int}} f_x\|_p^p \leqslant \frac{(C_p+1)}{K'} \frac{\|\nabla_H^{\sup} f\|_p^p}{\|f\|_p^p} \|f_x\|_p^p \right\}$$

has positive measure. Indeed, if it was not the case, then we would have

$$\int_{X_G} \|\nabla_G^{\mathrm{int}} f_x\|_p^p \; d\nu_G(x) \geqslant \frac{(C_p+1)}{K'} \frac{\|\nabla_H^{\sup} f\|_p^p}{\|f\|_p^p} \int_{X_G} \|f_x\|_p^p \; d\nu_G(x),$$

namely

$$\int_{X_G} \|\nabla_G^{\mathrm{int}} f_x\|_p^p \; d\nu_G(x) \geqslant (C_p+1)\|\nabla_H^{\sup} f\|_p^p$$

by Proposition 5.4, and this would contradict Proposition 5.8.

Given $\varepsilon > 0$, Markov's inequality and Proposition 5.3 imply

$$\nu_G\left(\left\{ x \in X_G \;:\; \lambda_G(\mathrm{supp} f_x) \geqslant \frac{1}{\varepsilon} K \lambda_H(\mathrm{supp} f) \right\}\right) \leqslant \varepsilon,$$

so assuming $\varepsilon < \mu(A)$, we find $x_0 \in A$ such that $\lambda_G(\mathrm{supp} f_{x_0}) \leqslant \frac{1}{\varepsilon} K \lambda_H(\mathrm{supp} f) \leqslant \frac{K}{\varepsilon} v$.

This finally gives

$$j^{\text{int}}_{p,G}\left(\frac{K}{\varepsilon}v\right) \geqslant \frac{\|f_{x_0}\|_p}{\|\nabla^{\text{int}}_G f_{x_0}\|_p} \geqslant \left(\frac{K'}{C_p+1}\right)^{1/p} \frac{\|f\|_p}{\|\nabla^{\sup}_H f\|_p}$$

and taking the supremum over f, we get

$$j^{\text{int}}_{p,G}\left(\frac{K}{\varepsilon}v\right) \geqslant \left(\frac{K'}{C_p+1}\right)^{1/p} j^{\sup}_{p,H}(v),$$

which completes the proof. □

Proof of Theorem 5.2. We assume without loss of generality that the measure equivalence coupling satisfies the properties provided by Lemma 3.1. Let v be a positive real number, let $f\colon H \to \mathbb{R}$ whose support has measure less than v and the associated maps f_x as explained in Section 5.1. We assume that $\|\nabla^{\sup}_H f\|_1 = 1$ (it is easy to see that the definition of the isoperimetric profile can be restricted to such functions). Let K, K', C_φ be the constants introduced in Propositions 5.3, 5.4 and 5.9. Since we have $\|\nabla^{\sup}_H f\|_1 = 1$, the constant C_φ does not depend on f. Moreover, this constant is finite by assumption and the comments after Definition 2.4 (namely S_G can be replaced by S_G^{-1}).

We claim that the set

$$A \coloneqq \left\{x \in X_G \,:\, \|\nabla^{\text{int}}_G f_x\|_1 \leqslant \frac{(C_\varphi+1)}{K'}\frac{2}{\varphi(2\|f\|_1)}\|f_x\|_1\right\}$$

has positive measure. Indeed, if it was not the case, then we would have

$$\int_{X_G} \|\nabla^{\text{int}}_G f_x\|_1 \, d\nu_G(x) \geqslant \frac{(C_\varphi+1)}{K'}\frac{2}{\varphi(2\|f\|_1)}\int_{X_G}\|f_x\|_1 \, d\nu_G(x),$$

namely

$$\int_{X_G} \|\nabla^{\text{int}}_G f_x\|_1 \, d\nu_G(x) \geqslant (C_\varphi+1)\frac{2\|f\|_1}{\varphi(2\|f\|_1)}$$

by Proposition 5.4, and this would contradict Proposition 5.9.

Given $\varepsilon > 0$, Markov's inequality and Proposition 5.3 imply

$$\nu_G\left(\left\{x \in X_G \,:\, \lambda_G(\text{supp} f_x) \geqslant \frac{1}{\varepsilon}K\lambda_H(\text{supp} f)\right\}\right) \leqslant \varepsilon,$$

so assuming $\varepsilon < \mu(A)$, we find $x_0 \in A$ such that $\lambda_G(\text{supp} f_{x_0}) \leqslant \frac{1}{\varepsilon}K\lambda_H(\text{supp} f) \leqslant \frac{K}{\varepsilon}v$.

This finally gives

$$j^{\text{int}}_{1,G}\left(\frac{K}{\varepsilon}v\right) \geqslant \frac{\|f_{x_0}\|_1}{\|\nabla^{\text{int}}_G f_{x_0}\|_1} \geqslant \frac{K'}{2(C_\varphi+1)}\varphi(2\|f\|_1)$$

and taking the supremum over the maps f satisfying $\|\nabla^{\sup}_H f\|_1 = 1$, we get

$$j^{\text{int}}_{p,G}\left(\frac{K}{\varepsilon}v\right) \geqslant \frac{K'}{2(C_\varphi+1)}\varphi(2j_{1,H}(v)) \geqslant \frac{K'}{2(C_\varphi+1)}\varphi(j^{\sup}_{1,H}(v)),$$

which completes the proof. □

6 Absence of quantitatively critical measure equivalence couplings

Theorem C states that, if there exists a (φ, L^0)-integrable measure equivalence coupling from G to H, then most of the time we have the following bound on φ:

$$\varphi \preccurlyeq j_{1,G} \circ j_{1,H}^{-1}, \tag{15}$$

and we would like to know if this upper bound can be reached. For discrete amenable groups, the first-named author proved that such upper bound cannot be achieved [Cor25, Theorem B], and we want to generalize it to the more general setting of locally compact compactly generated groups.

We need mild assumptions to get the inequality (15). First, we have to assume that $j_{1,H}$ is injective, so as to consider its inverse. We know that the isoperimetric profile is asymptotically equivalent to an injective map (see e.g. [Cor25, Remark 1.2]) so, in some sense, we may assume without loss of generality that the isoperimetric profile is injective. Secondly, the inequality $\varphi \circ j_{1,H} \preccurlyeq j_{1,G}$ means that there exists a constant $C > 0$ such that $\varphi \circ j_{1,H}(t) = O\left(j_{1,G}(Ct)\right)$, which means that $\varphi(t) = O\left(j_{1,G}(Cj_{1,H}^{-1}(t))\right)$. To get rid of this constant and obtain the desired composition $j_{1,G} \circ j_{1,H}^{-1}$, we will assume that the isoperimetric profile of G satisfies

$$\forall C > 0,\ j_{1,G}(Ct) = O\left(j_{1,G}(t)\right).$$

This condition already appeared in [Ers03; CD25]. There is no known example of a compactly generated group whose isoperimetric does not satisfy this condition.

Extending the proof of [Cor25, Theorem B] to the case of locally compact compactly generated amenable groups, we want to show Theorem D that we recall here.

Theorem 6.1. *Let G and H be non-discrete unimodular locally compact compactly generated groups. Assume that there exist a non-decreasing function f_G and an increasing function f_H satisfying $f_G(n) \simeq j_{1,G}(n)$, $f_H(n) \simeq j_{1,H}(n)$ and the following assumptions as $t \to +\infty$:*

$$f_G(t) = o\left(f_H(t)\right), \tag{16}$$

$$\forall C > 0,\ f_G(Ct) = O\left(f_G(t)\right), \tag{17}$$

$$\forall C > 0,\ f_G \circ f_H^{-1}(Ct) = O\left(f_G \circ f_H^{-1}(t)\right). \tag{18}$$

Then there is no $(f_G \circ f_H^{-1}, \mathrm{L}^0)$-integrable measure equivalence coupling from G to H.

About Theorem A dealing with volume growth, we prove a similar statement, as in [Cor25, Theorem D]. This is Theorem E that we recall here.

Theorem 6.2. *Let G and H be non-discrete unimodular locally compact compactly generated groups. Assume that there exist two increasing functions f_G and f_H satisfying $f_G(n) \simeq V_G(n)$, $f_H(n) \simeq V_H(n)$ and the following assumptions as $t \to +\infty$:*

$$f_G^{-1}(t) = o\left(f_H^{-1}(t)\right), \tag{19}$$

$$\forall C > 0,\ f_G^{-1}(Ct) = O\left(f_G^{-1}(t)\right), \tag{20}$$

$$\forall C > 0,\ f_G^{-1} \circ f_H(Ct) = O\left(f_G^{-1} \circ f_H(t)\right). \tag{21}$$

Then there is no $(f_G^{-1} \circ f_H, \mathrm{L}^0)$-integrable measure equivalence coupling from G to H.

Theorems 6.1 and 6.2 are direct consequences of the following crucial proposition.

Proposition 6.3. *Let G and H be locally compact compactly generated groups, let $G \curvearrowright (X, \nu)$ be a free p.m.p. action on a standard probability space. Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a sublinear and non-decreasing function. If an H-valued cocycle $\alpha\colon G \times X \to H$ is φ-integrable, then there exists a map $\psi\colon \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the following properties:*

- *$\varphi(t) \not\succcurlyeq \psi(t)$;*
- *the maps ψ and $t \mapsto t/\psi(t)$ are non-decreasing;*
- *ψ is subadditive;*
- *the cocycle α is ψ-integrable.*

We are now able to prove Theorems 6.1 and 6.2, using this proposition. The latter will be proved after.

Proof of Theorem 6.1. For $\varphi \coloneqq f_G \circ f_h^{-1}$, let us assume by contradiction that there exists a (φ, L^0)-integrable measure equivalence coupling from G to H. Then Proposition 6.3 implies that this coupling is also (ψ, L^0)-integrable with a map $\psi\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that ψ and $t \mapsto t/\psi(t)$ are non-decreasing, and $\psi \not\preccurlyeq f_G \circ f_H^{-1}$. But Theorem 6.1 implies $\psi \circ j_{1,H} \preccurlyeq j_{1,G}$, namely $\psi \preccurlyeq f_G \circ f_H^{-1}$ (see the end of the proof of [Cor25, Theorem B] for a justification), so we get a contradiction. ☐

Proof of Theorem 6.2. We proceed as in the last proof. Indeed, the map $\psi\colon \mathbb{R}_+ \to \mathbb{R}_+$ provided by Proposition 6.3 is also subadditive, an assumption of Theorem 6.2 which enables us to get a contradiction. ☐

We now proceed to the proof of Proposition 6.3. To this end, we have the following key lemma. In the case of finitely generated groups, this has been proved at the beginning of the proof of [Cor25, Lemma 3.2], which crucially uses finiteness of the generating sets. Here we manage to improve the proof in the non-discrete case.

Lemma 6.4. *Let G and H be locally compact compactly generated groups, and let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a map. Given a free p.m.p. action $G \curvearrowright (X, \nu)$ on a standard probability space, let $\alpha\colon G \times X \to H$ be a φ-integrable H-valued cocycle. Then there exist a measurable subset A of G and a sequence $(K_n)_{n \geqslant 1}$ of positive integers such that:*

- *$\lambda_G(A) > 0$ and A contains a countable dense subset of G;*
- *$K_n \underset{n \to +\infty}{\longrightarrow} +\infty$;*

- *for every $g \in A$, the series*

$$\sum_{n \geqslant 1} K_n \ \varphi(n) \ \nu(|c(g,\cdot)|_H = n)$$

converges.

Proof. Let D be a countable dense subset of G and let us enumerate its elements: $D = \{g_1, g_2, g_3, \ldots\}$. Let A_0 be a measurable subset of G such that $0 < \lambda_G(A_0) < +\infty$. First note that we have

$$\int_X \varphi(|c(g,x)|_H) d\nu(x) = \sum_{n \geqslant 0} \varphi(n) \ \nu(|c(g,\cdot)|_H = n)$$

for every $g \in G$, and these quantities converge by φ-integrability. We set

$$N(g,j) := \min \left\{ N \geqslant 1 \ : \ \sum_{n \geqslant N} \varphi(n) \ \nu(|c(g,\cdot)|_H = n) \leqslant \frac{1}{j^3} \right\}$$

for every $g \in G$ and every integer $j \geqslant 1$. Then we have

$$\lambda_G(\{g \in A_0 \mid N(g,j) > N) \underset{N \to +\infty}{\longrightarrow} 0,$$

so that there exists an increasing sequence $(N_j)_{j \geqslant 1}$ of positive integers such that

- $N_j \underset{j \to +\infty}{\longrightarrow} +\infty$;
- the series $\sum_{j \geqslant 1} \lambda_G(\{g \in A_0 \mid N(g,j) > N_j)$ converges;
- for every $j \geqslant 1$, for every $i \in \{1, \ldots, j\}$, $N(g_i, j) \leqslant N_j$.

First, given $i \geqslant 1$, we have

$$\sum_{n \geqslant N_j} \varphi(n) \ \nu(|c(g_i,\cdot)|_H = n) \leqslant \frac{1}{j^3}$$

for every $j \geqslant i$. Secondly, Borel-Cantelli lemma implies that for λ_G-almost every $g \in A_0$, let us say for every $g \in A_1$ with some $A_1 \subset A_0$ satisfying $\lambda_G(A_0 \backslash A_1) = 0$, we have

$$\sum_{n \geqslant N_j} \varphi(n) \ \nu(|c(g,\cdot)|_H = n) \leqslant \frac{1}{j^3}$$

for sufficiently large integers j.

To conclude, we have proved that for every $g \in D \cup A_1$, we have

$$\sum_{N_j \leqslant n \leqslant N_{j+1}-1} j \ \varphi(n) \ \nu(|c(g,\cdot)|_H = n) \leqslant \frac{1}{j^2}.$$

for every sufficiently large integer j. It now suffices to define $A = D \cup A_1$ and $K_n = j$ for every $n \in \{N_j, N_j + 1, \ldots, N_{j+1} - 1\}$ and we are done. ☐

Proof of Proposition 6.3. Let $(K_n)_{n \geqslant 1}$ be the sequence provided by Lemma 6.4, together with $A \subset G$ of positive Haar measure as in its statement, and let us denote by D a dense countable set contained in A. With the same method as in the proof of [Cor25, Lemma 3.2], we build a map $\psi \colon \mathbb{R}_+ \to \mathbb{R}_+$ from this sequence, which satisfies the following properties:

- ψ and $t \mapsto t/\psi(t)$ are non-decreasing;
- ψ is subbadditive;
- $\varphi(t) \not\succcurlyeq \psi(t)$;
- for every $g \in A$, $\|\alpha(g, .)\|_\psi$ is finite.

Given a compact generating set S_G of G, it now remains to prove that $\sup_{g \in S_G} \|\alpha(g, .)\|_\psi$ is finite. We use the same techniques as in [BFS13, Appendix A.2 (before Lemma A.1)]. For every $g \in G$, we abusively write

$$\|g\|_\psi \coloneqq \|\alpha(g, .)\|_\psi \in \mathbb{R} \cup \{+\infty\}.$$

Note that the cocycle identity, the subadditivity of ψ and the G-invariance of ν imply

$$\|g^{-1}\|_\psi = \|g\|_\psi \text{ and } \|gg'\|_\psi \leqslant \|g\|_\psi + \|g'\|_\psi \tag{22}$$

for every $g, g' \in G$. Denoting

$$E_t \coloneqq \{g \in G \mid \|g\|_\psi < t\}$$

for every $t > 0$, we have $\lambda_G\left(\bigcup_{t>0} E_t\right) \geqslant \lambda_G(A) > 0$, so there exists $t_0 > 0$ such that $\lambda_G(E_{t_0}) > 0$. This implies that $E_{t_0}^{-1} \cdot E_{t_0}$ is a neighbourhood of 1_G. But we also have $E_{t_0}^{-1} \cdot E_{t_0} = E_{t_0} \cdot E_{t_0} \subset E_{2t_0}$ using (22), so E_{2t_0} is a neighbourhood of 1_G. By density of D and compactness of S_G, there exists a finite subset F of D such that

$$S_G \subset \bigcup_{g \in F} g E_{2t_0}.$$

Once again by (22), we obtain for every $s \in S_G$:

$$\|s\|_\psi \leqslant 2t_0 + \max_{g \in F} \|g\|_\psi,$$

and so $\sup_{s \in S_G} \|s\|_\psi < +\infty$. This completes the proof. □

References

[Aus16] Tim Austin. "Integrable Measure Equivalence for Groups of Polynomial Growth". In: *Groups, Geometry, and Dynamics* 10.1 (2016), pp. 117–154. DOI: 10.4171/GGD/345.

[BFS13] Uri Bader, Alex Furman, and Roman Sauer. “Integrable Measure Equivalence and Rigidity of Hyperbolic Lattices”. In: *Inventiones mathematicae* 194.2 (2013), pp. 313–379. DOI: 10.1007/s00222-012-0445-9.

[BZ21] Jérémie Brieussel and Tianyi Zheng. “Speed of random walks, isoperimetry and compression of finitely generated groups”. In: *Annals of Mathematics* 193.1 (2021), pp. 1–105. DOI: 10.4007/annals.2021.193.1.1.

[CLM17] A. Carderi and F. Le Maître. “Orbit Full Groups for Locally Compact Groups”. In: *Transactions of the American Mathematical Society* 370.4 (2017), pp. 2321–2349. DOI: 10.1090/tran/6985.

[CDLH16] Yves Cornulier and Pierre De La Harpe. *Metric Geometry of Locally Compact Groups*. 1st ed. Vol. 25. EMS Tracts in Mathematics. EMS Press, 2016. ISBN: 978-3-03719-166-8 978-3-03719-666-3. DOI: 10.4171/166.

[Cor25] Corentin Correia. “On the Absence of Quantitatively Critical Measure Equivalence Couplings”. In: *Proceedings of the American Mathematical Society* (2025). DOI: 10.1090/proc/17291.

[CD25] Corentin Correia and Vincent Dumoncel. *Isoperimetric Profiles of Lamplighter-like Groups*. 2025. DOI: 10.48550/ARXIV.2506.13235.

[CS93] Thierry Coulhon and Laurent Saloff-Coste. “Isopérimétrie pour les groupes et les variétés.” In: *Revista Matemática Iberoamericana* 9.2 (1993), pp. 293–314.

[DKLMT22] T. Delabie, J. Koivisto, F. Le Maître, and R. Tessera. “Quantitative Measure Equivalence between Amenable Groups”. In: *Annales Henri Lebesgue* 5 (2022), pp. 1417–1487. ISSN: 2644-9463. DOI: 10.5802/ahl.155.

[DLIT25] Thiebout Delabie, Claudio Llosa Isenrich, and Romain Tessera. L^p *Measure Equivalence of Nilpotent Groups*. 2025. DOI: 10.48550/ARXIV.2505.17865.

[Dye59] H. A. Dye. “On Groups of Measure Preserving Transformations. I”. In: *American Journal of Mathematics* 81.1 (1959), pp. 119–159. DOI: 10.2307/2372852.

[Ers03] Anna Erschler. “On Isoperimetric Profiles of Finitely Generated Groups”. In: *Geometriae Dedicata* 100.1 (2003), pp. 157–171. DOI: 10.1023/A:1025849602376.

[EH24] Amandine Escalier and Camille Horbez. *Graph Products and Measure Equivalence: Classification, Rigidity, and Quantitative Aspects*. 2024. DOI: 10.48550/ARXIV.2401.04635.

[Fur11] A. Furman. “A Survey of Measured Group Theory”. In: *Geometry, Rigidity, and Group Actions*. Chicago and London: The University of Chicago Press, 2011, pp. 296–374. ISBN: 978-0-226-23788-6.

[Fur99] Alex Furman. “Gromov’s Measure Equivalence and Rigidity of Higher Rank Lattices”. In: *The Annals of Mathematics* 150.3 (1999), p. 1059. DOI: 10.2307/121062. JSTOR: 121062.

[Gro81] Michael Gromov. “Groups of polynomial growth and expanding maps (with an appendix by Jacques Tits)”. In: *Publications Mathématiques de l’IHÉS* 53 (1981), pp. 53–78.

[GH21] Vincent Guirardel and Camille Horbez. *Measure Equivalence Rigidity of* $\mathrm{Out}(F_N)$. 2021. DOI: 10.48550/ARXIV.2103.03696.

[HH22] Camille Horbez and Jingyin Huang. “Measure Equivalence Classification of Transvection-Free Right-Angled Artin Groups”. In: *Journal de l’École polytechnique — Mathématiques* 9 (2022), pp. 1021–1067. DOI: 10.5802/jep.199.

[HH24] Camille Horbez and Jingyin Huang. *Measure Equivalence Classification of Right-Angled Artin Groups: The Finite* Out *Classes*. 2024. DOI: 10.48550/ARXIV.2412.08560.

[Kid08] Yoshikata Kida. “The Mapping Class Group from the Viewpoint of Measure Equivalence Theory”. In: *Memoirs of the American Mathematical Society* 196.916 (2008), pp. 0–0. DOI: 10.1090/memo/0916.

[Kid10] Yoshikata Kida. “Measure Equivalence Rigidity of the Mapping Class Group”. In: *Annals of Mathematics* 171.3 (2010), pp. 1851–1901. DOI: 10.4007/annals.2010.171.1851.

[KKR21a] Juhani Koivisto, David Kyed, and Sven Raum. “Measure Equivalence and Coarse Equivalence for Unimodular Locally Compact Groups”. In: *Groups, Geometry, and Dynamics* 15.1 (2021), pp. 223–267. DOI: 10.4171/ggd/597.

[KKR21b] Juhani Koivisto, David Kyed, and Sven Raum. “Measure Equivalence for Non-Unimodular Groups”. In: *Transformation Groups* 26.1 (2021), pp. 327–346. DOI: 10.1007/s00031-021-09640-5.

[LNP25] Antonio López Neumann and Juan Paucar. *On Growth of Cocycles of Isometric Representations on* L^p*-Spaces*. 2025. DOI: 10.48550/ARXIV.2501.12808.

[OW80] Donald S. Ornstein and Benjamin Weiss. “Ergodic Theory of Amenable Group Actions. I: The Rohlin Lemma”. In: *Bulletin (New Series) of the American Mathematical Society* 2.1 (1980), pp. 161–164.

[Pau24] Juan Paucar. *Isoperimetric Profiles and Regular Embeddings of Locally Compact Groups*. 2024. DOI: 10.48550/ARXIV.2402.16787.

[Pit95] Christophe Pittet. “Følner sequences in polycyclic groups”. In: *Revista Matemática Iberoamericana* 11.3 (1995), pp. 675–685. DOI: 10.4171/rmi/189.

[Pit00] Christophe Pittet. "The isoperimetric profile of homogeneous Riemannian manifolds". In: *Journal of Differential Geometry* 54.2 (2000). DOI: 10.4310/jdg/1214341647.

[Sha04] Yehuda Shalom. "Harmonic Analysis, Cohomology, and the Large-Scale Geometry of Amenable Groups". In: *Acta Mathematica* 192.2 (2004), pp. 119–185. DOI: 10.1007/BF02392739.

[Tes08] Romain Tessera. "Large-Scale Sobolev Inequalities on Metric Measure Spaces and Applications". In: *Revista Matemática Iberoamericana* 24.3 (2008), pp. 825–864. DOI: 10.4171/rmi/557.

[Tes13] Romain Tessera. "Isoperimetric profiles and random walks on locally compact solvable groups". In: *Revista Matematica Iberoamericana* 29.2 (2013), pp. 715–737. DOI: 10.4171/RMI/736.

[Ver73] Anatole Vershik. *Countable groups close to finite ones, Appendix to the Russian translation of the book "Invariant means on topological groups and their applications" by F.P. Greenleaf*. Mir, 1973, pp. 112–135.

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